Stones for Bread: A Classical Reading of Intuitionism

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7th Annual IAL Conference, Tehran 1398

BHK Interpretation

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- $\bot$ has no proof.

Interpreting "*Proofs*"

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- Instead of defining classical proofs, he reinvent the system **S4** as an abstract calculus for classical provability:
  - Axiom **K**: $\Box(A \to B) \to (\Box A \to \Box B)$,
  - Axiom **T**: $\Box A \to A$,
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    - **MP**: If A and $A \to B$ are provable then B is also provable,
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  If we interpret $\Box$ as *informal provability* then all the axioms and rules are valid.

A Formalization for BHK Interpretation via Classical Proofs

Then he interpreted the BHK interpretation as the following translation from **IPC** to **S4**. Read A^b as "*the existence of a proof for A*":

- $p^b = \Box p$ and $\bot^b = \Box\bot$
- $(A \circ B)^b = A^b \circ B^b$ for $\circ \in \{\wedge, \vee\}$
- $(A \to B)^b = \Box(A^b \to B^b)$

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The soundness and completeness of this classical interpretation of proof is established via the soundness and completeness for the translation, i.e., **IPC** $\vdash A$ iff **S4** $\vdash A^b$, meaning **IPC** $\vdash A$ iff "*A has a proof*".

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Gödel's 1933 Problem

Is it possible to formalize this *informal provability interpretation* using some *concrete* classical proofs?

The Simplest Approach

The most natural attempt is fixing a strong enough r.e. theory T and interpret $\Box$ as a natural provability predicate for the theory T.

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Where is the clash between the previous interpretation and the intuitive interpretation?

In the formula $\Box(\Box\bot \to \bot)$, the inner box refers to the provability in a theory T, but the outer box refers to the provability in the meta-theory of T which is not necessarily equal to T itself.

A More Sophisticated Approach

In this sense the natural interpretation of a modal formula needs:

- A model M capturing the real world and,
- A hierarchy of theories $\{T_n\}_{n=0}^{\infty}$ capturing the whole hierarchy of theories, meta-theories, meta-meta-theories and so on.

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Definition

A provability model is a pair $(M, \{T_n\}_{n=0}^{\infty})$ where M is a model of $I\Sigma_1$ and $\{T_n\}_{n=0}^{\infty}$ is a hierarchy of arithmetical r.e. theories such that for any n, $I\Sigma_1 \subseteq T_n \subseteq T_{n+1}$ provably in $I\Sigma_1$.

A provability model $(M, \{T_n\}_{n=0}^{\infty})$ is called reflexive if for any n, M thinks that T_n is sound and $T_{n+1} \vdash \mathrm{Rfn}(T_n)$ i.e., $M \vDash \mathsf{Pr}_n(A) \to A$ and $M \vDash \mathsf{Pr}_{n+1}(\mathsf{Pr}_n(A) \to A)$, for every n and A. We will denote this class by **Ref**.

Some Classes of Provability Models

In the following we will define some useful classes of provability models:

Definition

- We denote the class of all provability models by **PrM**.
- A provability model $(M, \{T_n\}_{n=0}^{\infty})$ is called consistent if for any n, M thinks that T_n is consistent and $T_{n+1} \vdash \mathrm{Cons}(T_n)$. We will denote this class by **Cons**.
- A provability model $(M, \{T_n\}_{n=0}^{\infty})$ is called reflexive if for any n, M thinks that T_n is sound and $T_{n+1} \vdash \mathrm{Rfn}(T_n)$. We will denote this class by **Ref**.
- A provability model $(M, \{T_n\}_{n=0}^{\infty})$ is called constant if for any n, M thinks that $T_n = T_0$. We will denote this class by **Cst**.
- A provability model $(M, \{T_n\}_{n=0}^{\infty})$ is called sound constant if for any n, M thinks that T_n is sound and $T_n = T_0$. We will denote this class by **sCst**.

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Definition

Let w be a witness for A and σ an arithmetical substitution which assigns an arithmetical sentence to a propositional variable. And also let $(M, \{T_n\}_{n=0}^{\infty})$ be a provability model. By $A^\sigma(w)$ we mean an arithmetical sentence which results by substituting the variables by the values of σ and interpreting any box as the provability predicate of T_n if the corresponding number in the witness for this box was n. The interpretation of boolean connectives are themselves.

Example

Let $(M, \{T_n\}_{n=0}^{\infty})$ be a provability model. Then the formula $A = \Box(p \to q) \to (\Box p \to \Box q)$ is true in this model. It is enough to pick the witness $w = (0, 0, 0)$. Then the interpretation of the formula under the arithmetical interpretation σ is $A^\sigma(w) = \mathrm{Pr}_{T_0}(p^\sigma \to q^\sigma) \to (\mathrm{Pr}_{T_0}(p^\sigma) \to \mathrm{Pr}_{T_0}(q^\sigma))$ which is true in M, because $I\Sigma_1$ proves the distributivity of provability predicates over the implication and $M \vDash I\Sigma_1$.

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Example

Let $(M, \{T_n\}_{n=0}^{\infty})$ be a reflexive provability model. Then the formula $A = \Box(\Box p \to p)$ is true in this model. It is enough to pick the witness $w = (1, 0)$. Then the interpretation of the formula under the arithmetical interpretation σ is $A^\sigma(w) = \mathrm{Pr}_{T_1}(\mathrm{Pr}_{T_0}(p^\sigma) \to p^\sigma)$ which is true in M.

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A Conjectured Soundness-Completeness Theorem

S4 $\vdash A$ iff there exists a witness for A such that all arithmetical interpretations of A in all reflexive models hold, i.e.,

$$\mathbf{S4} \vdash A \iff \exists w \forall \sigma \forall (M, \{T_n\}_{n=0}^{\infty}) \in \mathbf{Ref}\ M \vDash A^\sigma(w).$$

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The $\exists w$ is based on the assumption that there were valid indices by which we informally argued but now we have forgotten them.

The Herbrand Phenomenon

Unfortunately, this conjecture is false. For instance while the formula $\neg\Box(\neg\Box p \wedge p)$ is provable in **S4**, it has no witness that works for all reflexive provability models.

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Assume $\Box(\neg\Box p \wedge p)$. Firstly, by **T**, we have $\neg\Box p \wedge p$ and hence $\neg\Box p$. Secondly, by **K**, box commutes with conjunction and hence we have $\Box\neg\Box p$ and $\Box p$. Hence, we reach a contradiction.

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The reason is the different roles that one box can play. Our interpretation assumes there was only one index for any box that we have forgotten and we want to remember. This is not true.

The Herbrand Phenomenon

Think about the formula $\neg\Box_2(\neg\Box_1 p \wedge p) \vee \neg\Box_1(\neg\Box_0 p \wedge p)$. This is valid in reflexive models. Because if we have both $\Box_2(\neg\Box_1 p \wedge p)$ and $\Box_1(\neg\Box_0 p \wedge p)$, then from the first we have $\neg\Box_1 p$ and from the second we have $\Box_1 p$ which leads to a contradiction.

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If we forget the indices, then we have $\neg\Box(\neg\Box p \wedge p) \vee \neg\Box(\neg\Box p \wedge p)$ which is equivalent to $\neg\Box(\neg\Box p \wedge p)$. But based on our interpretation, when we want to remember the index, it can be either $\neg\Box_2(\neg\Box_1 p \wedge p)$ or $\neg\Box_1(\neg\Box_0 p \wedge p)$, which is different from their disjunction.

Expansions

To capture these different roles we introduce expansions. They are similar to expansions in the generalized Herbrand's theorem.

Definition

$E(A)$, the set of all expansions of A, is inductively defined as follows:

- If A is an atom, $E(A) = \{A\}$.
- If $A = B \circ C$, then $E(A) = \{D \circ E \mid D \in E(B) \text{ and } E \in E(C)\}$ for $\circ \in \{\wedge, \vee, \rightarrow\}$.
- If $A = \neg B$, then $E(A) = \{\neg D \mid D \in E(B)\}$.
- If $A = \Box B$, then $E(A) = \{\Box \bigvee_{i=1}^{k} D_i \mid \forall 1 \leq i \leq k,\ D_i \in E(B)\}$.

Informally speaking, an expansion of a formula A is a formula constructed by replacing any formula after a box with disjunctions of the expansions of the formula. For instance, $\Box(\Box p \vee \Box p)$ is an expansion for $\Box\Box p$.

The Main Theorem, Positive Part

Soundness-Completeness Theorem

S4 $\vdash A$ iff there exist finite number of expansions of A like $B_1, \ldots, B_k$, a witness for $\bigvee_{i=1}^{k} B_i$ such that all arithmetical interpretations of $\bigvee_{i=1}^{k} B_i$ in all reflexive models hold, i.e.,

$$\mathbf{S4} \vdash A \Longleftrightarrow \exists w \exists B_1, \ldots B_k \forall \sigma \forall (M, \{T_n\}_{n=0}^{\infty}) \in \mathbf{Ref}\ M \vDash (\bigvee_{i=1}^{k} B_i)^{\sigma}(w).$$

The same is also true for the pairs $(\mathbf{K4}, \mathbf{PrM})$, $(\mathbf{KD4}, \mathbf{Cons})$, $(\mathbf{GL}, \mathbf{Cst})$ and $(\mathbf{GLS}, \mathbf{sCst})$.

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Proof.

For soundness use the cut-free system for the logic. For completeness, use a modification of Solovay's technique. □

The Main Theorem, Negative Part

No-Models Theorem

There is no provability model for any extension of **KD45**, i.e., for any $(M, \{T_n\}_{n=0}^{\infty})$ there exists A such that $\mathbf{KD45} \vdash A$ and for any finite number of expansions of A like $B_1, \ldots, B_k$, any witness for $\bigvee_{i=1}^{k} B_i$ there exists an arithmetical substitution for $\bigvee_{i=1}^{k} B_i$ such that $M \nvDash (\bigvee_{i=1}^{k} B_i)^{\sigma}(w)$.

Proof.

Sketch. Let us prove the theorem for **S5** and $M = \mathbb{N}$. If $(\mathbb{N}, \{T_n\}_{n=0}^{\infty})$ validates **T**, **4** and **5**, it implies the existence of some n such that:

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- **4**: If $T_n \vdash A$ then $T_{n+1} \vdash \mathrm{Pr}_n(A)$,
- **5**: If $T_n \nvdash A$ then $T_{n+1} \vdash \neg \mathrm{Pr}_n(A)$,
- **T**: It is impossible to have both $T_{n+1} \vdash \mathrm{Pr}_n(A)$ and $T_{n+1} \vdash \neg \mathrm{Pr}_n(A)$.

Therefore, since T_{n+1} is r.e., the provability of T_n is decidable. □

Modal Characterizations

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Modal	Provability Models
K4	All Models
KD4	Consistent Models
S4	Reflexive Models
GL	Constant Models
Above **KD45**	No Models

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A provability model $(M, \{T_n\}_{n=0}^{\infty})$ is called a BHK model if $M \vDash \neg \mathrm{Pr}_{n+1}(\mathrm{Pr}_n(\bot))$, for any n.

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This means that not only all T_n's are consistent but also T_{n+1} can not think otherwise.

For any class $\mathcal{C}$ of provability models, by $\mathcal{C}^b$ we mean the class of all BHK models in $\mathcal{C}$.

Combining our provability interpretation with Gödel's translation, we will have a formalization for the BHK interpretation via classical proofs:

Soundness-Completeness Theorem

IPC $\vdash A$ iff there exist finite number of expansions of A^b like B_1, ..., B_k, a witness for $\bigvee_{i=1}^{k} B_i$ such that all arithmetical interpretations of $\bigvee_{i=1}^{k} B_i$ in all reflexive models hold, i.e.,

$$\mathbf{IPC} \vdash A \iff \exists w \exists B_1, \ldots B_k \forall \sigma \forall (M, \{T_n\}_{n=0}^{\infty}) \in \mathbf{Ref}\ M \vDash (\bigvee_{i=1}^{k} B_i)^{\sigma}(w).$$

Classical Reading of Intuitionism

Combining our provability interpretation with Gödel's translation, we will have a formalization for the BHK interpretation via classical proofs:

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The same is also true for the pairs $(\mathbf{BPC}, \mathbf{PrM}^b)$, $(\mathbf{EBPC}, \mathbf{Cons})$ and $(\mathbf{FPL}, \mathbf{Cst}^b)$.

No BHK interpretation for Classical logic

No-Models Theorem

Let $(M, \{T_n\}_{n=0}^{\infty})$ be a BHK model. Then it does not validate **CPC**, i.e., there exists A such that $\mathbf{CPC} \vdash A$ and for any finite number of expansions of A^b like $B_1, \ldots, B_k$, any witness for $\bigvee_{i=1}^{k} B_i$, there exists an arithmetical substitution for $\bigvee_{i=1}^{k} B_i$ such that $M \nvDash (\bigvee_{i=1}^{k} B_i)^{\sigma}(w)$.

All Characterizations

Modal	**Propositional**	**Provability Models**
K4	**BPC**	All Models
KD4	**EBPC**	Consistent Models
S4	**IPC**	Reflexive Models
GL	**FPL**	Constant Models
Above **KD45**	**CPC**	No Models

A Philosophical Consequence

In all propositional results the Gödel's translation (BHK interpretation) is fixed. Therefore, the result suggests that believing only in BHK interpretation, there could be different yet equally valid *intuitionistic logics* rather than *one* intuitionitic logic. The difference between these logics is in the ontological commitments that we put on our meta-theories:

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- **IPC** is *intuitionistic logic* if we believe in soundness of our theories, provably.
- **FPL** is *intuitionistic logic* if we believe in one theory for all the meta-levels.

Thank you for your attention!